

Iterated crossed products *

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Dedicated to Professor Constantin Năstăsescu on the occasion of his 70th birthday

Abstract

We define a "mirror version" of Brzeziński's crossed product and we prove that, under certain circumstances, a Brzeziński crossed product $D \otimes_{R,\sigma} V$ and a mirror version $W \overline{\otimes}_{P,\nu} D$ may be iterated, obtaining an algebra structure on $W \otimes D \otimes V$. Particular cases of this construction are the iterated twisted tensor product of algebras and the quasi-Hopf two-sided smash product.

Introduction

If A and B are associative unital algebras and $R : B \otimes A \rightarrow A \otimes B$ is a linear map satisfying certain axioms (such an R is called a twisting map) then $A \otimes B$ becomes an associative unital algebra with a multiplication defined in terms of R and the multiplications of A and B ; this algebra structure is denoted by $A \otimes_R B$ and called the twisted tensor product of A and B afforded by R (cf. [4], [9]). There exist many concrete examples of twisted tensor products, for instance the Hopf smash product and other kinds of products arising in Hopf algebra theory.

In [6], two types of general results have been proved for twisted tensor products. One was called "invariance under twisting" (since it was directly inspired by the invariance under twisting of the Hopf smash product). The other one is the fact that twisted tensor products may be iterated. More precisely, given three twisted tensor products $A \otimes_{R_1} B$, $B \otimes_{R_2} C$ and $A \otimes_{R_3} C$, it has been proved that a sufficient condition for being able to define certain twisting maps $T_1 : C \otimes (A \otimes_{R_1} B) \rightarrow (A \otimes_{R_1} B) \otimes C$ and $T_2 : (B \otimes_{R_2} C) \otimes A \rightarrow A \otimes (B \otimes_{R_2} C)$ associated to R_1, R_2, R_3 and ensuring that the algebras $A \otimes_{T_2} (B \otimes_{R_2} C)$ and $(A \otimes_{R_1} B) \otimes_{T_1} C$ are equal (this algebra is called the iterated twisted tensor product), can be given in terms of the maps R_1, R_2, R_3 , namely, they have to satisfy the braid relation $(id_A \otimes R_2) \circ (R_3 \otimes id_B) \circ (id_C \otimes R_1) = (R_1 \otimes id_C) \circ (id_B \otimes R_3) \circ (R_2 \otimes id_A)$.

On the other hand, there exist important examples of products of algebras that are not twisted tensor products, a prominent example being the classical Hopf crossed product. In [1], Brzeziński introduced a very general construction, called crossed product, containing as particular cases twisted tensor products of algebras as well as classical Hopf crossed products. Given an associative unital algebra A , a vector space V endowed with a distinguished element 1_V and two linear maps $\sigma : V \otimes V \rightarrow A \otimes V$ and $R : V \otimes A \rightarrow A \otimes V$ satisfying certain conditions,

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Brzeziński's crossed product is a certain associative unital algebra structure on $A \otimes V$, denoted in what follows by $A \otimes_{R,\sigma} V$. Another example of a crossed product is the smash product $H \# B$ between a quasi-bialgebra H and a right H -module algebra B , cf. [3] (since in general B is not associative, $H \# B$ in general is not a twisted tensor product of algebras).

In [7] we have proved a result of the type invariance under twisting for crossed products, that arose as a common generalization of the invariance under twisting for twisted tensor products of algebras and the invariance under twisting for quasi-Hopf smash products.

In this paper we will study iterated crossed products. Our motivating example was the so-called quasi-Hopf two-sided smash product $A \# H \# B$ from [3], where H is a quasi-bialgebra and A (respectively B) is a left (respectively right) H -module algebra. We wanted to express $A \# H \# B$ as some sort of iterated product between the two smash products $A \# H$ and $H \# B$. As we have seen before, $H \# B$ is a Brzeziński crossed product. We needed to express also $A \# H$ as some sort of crossed product. It turns out that there exists a "mirror version" of Brzeziński's crossed product, which we denote by $W \overline{\otimes}_{P,\nu} D$ (where D is an associative algebra, W is a vector space and P, ν are certain maps), and $A \# H$ is an example of such a crossed product. Our result, that contains as particular cases both the above $A \# H \# B$ and the result about iterated twisted tensor products of algebras, may be formulated as follows: if $W \overline{\otimes}_{P,\nu} D$ and $D \otimes_{R,\sigma} V$ are two crossed products and $Q : V \otimes W \rightarrow W \otimes D \otimes V$ is a linear map satisfying certain conditions, then one can define certain maps $\overline{\sigma}, \overline{R}, \overline{\nu}, \overline{P}$ such that we have the crossed products $(W \overline{\otimes}_{P,\nu} D) \otimes_{\overline{R},\overline{\sigma}} V$ and $W \overline{\otimes}_{\overline{P},\overline{\nu}} (D \otimes_{R,\sigma} V)$ that are moreover equal as algebras (this algebra structure is called the iterated crossed product). Moreover, this result admits a certain converse. Finally, we prove that a certain construction introduced in [8] is an example of an iterated crossed product.

1 Preliminaries

We work over a commutative field k . All algebras, linear spaces etc. will be over k ; unadorned \otimes means \otimes_k . By "algebra" we always mean an associative unital algebra. The multiplication of an algebra A is denoted by μ_A or simply μ when there is no danger of confusion, and we usually denote $\mu_A(a \otimes a') = aa'$ for all $a, a' \in A$.

We recall from [4], [9] that, given two algebras A, B and a k -linear map $R : B \otimes A \rightarrow A \otimes B$, with notation $R(b \otimes a) = a_R \otimes b_R$, for $a \in A, b \in B$, satisfying the conditions $a_R \otimes 1_R = a \otimes 1$, $1_R \otimes b_R = 1 \otimes b$, $(aa')_R \otimes b_R = a_R a'_R \otimes b_R$, $a_R \otimes (bb')_R = a_{R_r} \otimes b_r b'_R$, for all $a, a' \in A$ and $b, b' \in B$ (where r and R are two different indices), if we define on $A \otimes B$ a new multiplication, by $(a \otimes b)(a' \otimes b') = aa'_R \otimes b_R b'$, then this multiplication is associative with unit $1 \otimes 1$. In this case, the map R is called a **twisting map** between A and B and the new algebra structure on $A \otimes B$ is denoted by $A \otimes_R B$ and called the **twisted tensor product** of A and B afforded by the map R .

We recall from [1] the construction of Brzeziński's crossed product:

Proposition 1.1 ([1]) *Let $(A, \mu, 1_A)$ be an (associative unital) algebra and V a vector space equipped with a distinguished element $1_V \in V$. Then the vector space $A \otimes V$ is an associative algebra with unit $1_A \otimes 1_V$ and whose multiplication has the property that $(a \otimes 1_V)(b \otimes v) = ab \otimes v$, for all $a, b \in A$ and $v \in V$, if and only if there exist linear maps $\sigma : V \otimes V \rightarrow A \otimes V$ and $R : V \otimes A \rightarrow A \otimes V$ satisfying the following conditions:*

$$R(1_V \otimes a) = a \otimes 1_V, \quad R(v \otimes 1_A) = 1_A \otimes v, \quad \forall a \in A, v \in V, \quad (1.1)$$

$$\sigma(1_V, v) = \sigma(v, 1_V) = 1_A \otimes v, \quad \forall v \in V, \quad (1.2)$$

$$R \circ (id_V \otimes \mu) = (\mu \otimes id_V) \circ (id_A \otimes R) \circ (R \otimes id_A), \quad (1.3)$$

$$\begin{aligned} & (\mu \otimes id_V) \circ (id_A \otimes \sigma) \circ (R \otimes id_V) \circ (id_V \otimes \sigma) \\ &= (\mu \otimes id_V) \circ (id_A \otimes \sigma) \circ (\sigma \otimes id_V), \end{aligned} \quad (1.4)$$

$$\begin{aligned} & (\mu \otimes id_V) \circ (id_A \otimes \sigma) \circ (R \otimes id_V) \circ (id_V \otimes R) \\ &= (\mu \otimes id_V) \circ (id_A \otimes R) \circ (\sigma \otimes id_A). \end{aligned} \quad (1.5)$$

If this is the case, the multiplication of $A \otimes V$ is given explicitly by

$$\mu_{A \otimes V} = (\mu_2 \otimes id_V) \circ (id_A \otimes id_A \otimes \sigma) \circ (id_A \otimes R \otimes id_V),$$

where $\mu_2 = \mu \circ (id_A \otimes \mu) = \mu \circ (\mu \otimes id_A)$. We denote by $A \otimes_{R,\sigma} V$ this algebra structure and call it the **crossed product** afforded by the data (A, V, R, σ) .

If $A \otimes_{R,\sigma} V$ is a crossed product, we introduce the following Sweedler-type notation:

$$\begin{aligned} R : V \otimes A &\rightarrow A \otimes V, & R(v \otimes a) &= a_R \otimes v_R, \\ \sigma : V \otimes V &\rightarrow A \otimes V, & \sigma(v \otimes v') &= \sigma_1(v, v') \otimes \sigma_2(v, v'), \end{aligned}$$

for all $v, v' \in V$ and $a \in A$. With this notation, the multiplication of $A \otimes_{R,\sigma} V$ reads

$$(a \otimes v)(a' \otimes v') = aa'_R \sigma_1(v_R, v') \otimes \sigma_2(v_R, v'), \quad \forall a, a' \in A, v, v' \in V.$$

A twisted tensor product is a particular case of a crossed product (cf. [5]), namely, if $A \otimes_R B$ is a twisted tensor product of algebras then $A \otimes_R B = A \otimes_{R,\sigma} B$, where $\sigma : B \otimes B \rightarrow A \otimes B$ is given by $\sigma(b \otimes b') = 1_A \otimes bb'$, for all $b, b' \in B$.

Remark 1.2 The conditions (1.3), (1.4) and (1.5) for R, σ may be written in Sweedler-type notation respectively as

$$(aa')_R \otimes v_R = a_R a'_r \otimes v_{R_r}, \quad (1.6)$$

$$\begin{aligned} & \sigma_1(y, z)_R \sigma_1(x_R, \sigma_2(y, z)) \otimes \sigma_2(x_R, \sigma_2(y, z)) \\ &= \sigma_1(x, y) \sigma_1(\sigma_2(x, y), z) \otimes \sigma_2(\sigma_2(x, y), z), \end{aligned} \quad (1.7)$$

$$a_{R_r} \sigma_1(v_r, v'_R) \otimes \sigma_2(v_r, v'_R) = \sigma_1(v, v') a_R \otimes \sigma_2(v, v')_R, \quad (1.8)$$

for all $a, a' \in A$ and $x, y, z, v, v' \in V$, where r is another copy of R .

2 The main result and its particular cases

We begin by defining the mirror version of Brzeziński's crossed product; the proof is similar to the one in [1] and will be omitted.

Theorem 2.1 Let $(B, \mu, 1_B)$ be an (associative unital) algebra and W a vector space equipped with a distinguished element $1_W \in W$. Then the vector space $W \otimes B$ is an associative algebra with unit $1_W \otimes 1_B$ and whose multiplication has the property that $(w \otimes b)(1_W \otimes b') = w \otimes bb'$, for all $b, b' \in B$ and $w \in W$, if and only if there exist linear maps $\nu : W \otimes W \rightarrow W \otimes B$ and $P : B \otimes W \rightarrow W \otimes B$ satisfying the following conditions:

$$P(b \otimes 1_W) = 1_W \otimes b, \quad P(1_B \otimes w) = w \otimes 1_B, \quad \forall b \in B, w \in W, \quad (2.1)$$

$$\nu(w \otimes 1_W) = \nu(1_W \otimes w) = w \otimes 1_B, \quad \forall w \in W, \quad (2.2)$$

$$P \circ (\mu \otimes id_W) = (id_W \otimes \mu) \circ (P \otimes id_B) \circ (id_B \otimes P), \quad (2.3)$$

$$\begin{aligned} & (id_W \otimes \mu) \circ (\nu \otimes id_B) \circ (id_W \otimes P) \circ (\nu \otimes id_W) \\ &= (id_W \otimes \mu) \circ (\nu \otimes id_B) \circ (id_W \otimes \nu), \end{aligned} \quad (2.4)$$

$$\begin{aligned} & (id_W \otimes \mu) \circ (\nu \otimes id_B) \circ (id_W \otimes P) \circ (P \otimes id_W) \\ &= (id_W \otimes \mu) \circ (P \otimes id_B) \circ (id_B \otimes \nu). \end{aligned} \quad (2.5)$$

If this is the case, the multiplication of $W \otimes B$ is given explicitly by

$$\mu_{W \otimes B} = (id_W \otimes \mu_2) \circ (\nu \otimes id_B \otimes id_B) \circ (id_W \otimes P \otimes id_B),$$

where $\mu_2 = \mu \circ (id_B \otimes \mu) = \mu \circ (\mu \otimes id_B)$. We denote by $W \overline{\otimes}_{P,\nu} B$ this algebra structure and call it the **crossed product** afforded by the data (W, B, P, ν) .

If $W \overline{\otimes}_{P,\nu} B$ is a crossed product, we use the Sweedler-type notation

$$\begin{aligned} P : B \otimes W &\rightarrow W \otimes B, & P(b \otimes w) &= w_P \otimes b_P, \\ \nu : W \otimes W &\rightarrow W \otimes B, & \nu(w \otimes w') &= \nu_1(w, w') \otimes \nu_2(w, w'), \end{aligned}$$

for all $w, w' \in W$ and $b \in B$. With this notation, the multiplication of $W \overline{\otimes}_{P,\nu} B$ reads

$$(w \otimes b)(w' \otimes b') = \nu_1(w, w'_P) \otimes \nu_2(w, w'_P) b_P b', \quad \forall b, b' \in B, w, w' \in W.$$

If $A \otimes_R B$ is a twisted tensor product of algebras, then $A \otimes_R B = A \overline{\otimes}_{R,\nu} B$, where $\nu : A \otimes A \rightarrow A \otimes B$, $\nu(a \otimes a') = aa' \otimes 1_B$ for all $a, a' \in A$.

Remark 2.2 The conditions (2.3), (2.4) and (2.5) may be written down respectively as

$$w_P \otimes (bb')_P = w_{P_p} \otimes b_p b'_P, \quad (2.6)$$

$$\begin{aligned} & \nu_1(\nu_1(w, w'), w''_P) \otimes \nu_2(\nu_1(w, w'), w''_P) \nu_2(w, w')_P \\ &= \nu_1(w, \nu_1(w', w'')) \otimes \nu_2(w, \nu_1(w', w'')) \nu_2(w', w''), \end{aligned} \quad (2.7)$$

$$\nu_1(w_P, w'_P) \otimes \nu_2(w_P, w'_P) b_{P_p} = \nu_1(w, w')_P \otimes b_P \nu_2(w, w'), \quad (2.8)$$

for all $w, w', w'' \in W$ and $b, b' \in B$, where p is another copy of P .

Now we can prove that under certain circumstances the two versions of crossed products may be iterated:

Theorem 2.3 Let $W \overline{\otimes}_{P,\nu} D$ and $D \otimes_{R,\sigma} V$ be two crossed products. Assume that we have a linear map $Q : V \otimes W \rightarrow W \otimes D \otimes V$, with notation $Q(v \otimes w) = Q_W(v, w) \otimes Q_D(v, w) \otimes Q_V(v, w)$, for all $v \in V$, $w \in W$, such that $Q(1_V \otimes w) = w \otimes 1_D \otimes 1_V$, $Q(v \otimes 1_W) = 1_W \otimes 1_D \otimes v$, for all $v \in V$, $w \in W$, and the following conditions are satisfied:

$$\begin{aligned} & (id_W \otimes \mu_D \otimes id_V) \circ (id_W \otimes id_D \otimes R) \circ (Q \otimes id_D) \circ (id_V \otimes P) \\ &= (id_W \otimes \mu_D \otimes id_V) \circ (P \otimes id_D \otimes id_V) \circ (id_D \otimes Q) \circ (R \otimes id_W), \end{aligned} \quad (2.9)$$

$$\begin{aligned} & (id_W \otimes \mu_D \otimes id_V) \circ (id_W \otimes id_D \otimes R) \circ (Q \otimes id_D) \circ (id_V \otimes \nu) \\ &= (id_W \otimes \mu_D \otimes id_V) \circ (\nu \otimes \mu_D \otimes id_V) \circ (id_W \otimes P \otimes id_D \otimes id_V) \\ &\quad \circ (id_W \otimes id_D \otimes Q) \circ (Q \otimes id_W), \end{aligned} \quad (2.10)$$

$$(id_W \otimes \mu_D \otimes id_V) \circ (P \otimes id_D \otimes id_V) \circ (id_D \otimes Q) \circ (\sigma \otimes id_W)$$

$$\begin{aligned}
&= (id_W \otimes \mu_D \otimes id_V) \circ (id_W \otimes \mu_D \otimes \sigma) \circ (id_W \otimes id_D \otimes R \otimes id_V) \\
&\quad \circ (Q \otimes id_D \otimes id_V) \circ (id_V \otimes Q).
\end{aligned} \tag{2.11}$$

(i) Define the maps

$$\begin{aligned}
\bar{\sigma} &: V \otimes V \rightarrow (W \bar{\otimes}_{P,\nu} D) \otimes V, \\
\bar{\sigma}(v \otimes v') &= (1_W \otimes \sigma_1(v, v')) \otimes \sigma_2(v, v'), \quad \forall v, v' \in V, \\
\bar{R} &: V \otimes (W \bar{\otimes}_{P,\nu} D) \rightarrow (W \bar{\otimes}_{P,\nu} D) \otimes V, \\
\bar{R} &= (id_W \otimes \mu_D \otimes id_V) \circ (id_W \otimes id_D \otimes R) \circ (Q \otimes id_D)
\end{aligned}$$

(i.e. for all $v \in V, w \in W, d \in D$ we have $\bar{R}(v \otimes w \otimes d) = Q_W(v, w) \otimes Q_D(v, w) d_R \otimes Q_V(v, w)_R$). Then we have a crossed product $(W \bar{\otimes}_{P,\nu} D) \otimes_{\bar{R}, \bar{\sigma}} V$.

(ii) Define the maps

$$\begin{aligned}
\bar{\nu} &: W \otimes W \rightarrow W \otimes (D \otimes_{R,\sigma} V), \\
\bar{\nu}(w \otimes w') &= \nu_1(w, w') \otimes (\nu_2(w, w') \otimes 1_V), \quad \forall w, w' \in W, \\
\bar{P} &: (D \otimes_{R,\sigma} V) \otimes W \rightarrow W \otimes (D \otimes_{R,\sigma} V), \\
\bar{P} &= (id_W \otimes \mu_D \otimes id_V) \circ (P \otimes id_D \otimes id_V) \circ (id_D \otimes Q)
\end{aligned}$$

(i.e. for all $v \in V, w \in W, d \in D$ we have $\bar{P}(d \otimes v \otimes w) = Q_W(v, w)_P \otimes d_P Q_D(v, w) \otimes Q_V(v, w)$). Then we have a crossed product $W \bar{\otimes}_{\bar{P}, \bar{\nu}} (D \otimes_{R,\sigma} V)$.

(iii) We have an algebra isomorphism $(W \bar{\otimes}_{P,\nu} D) \otimes_{\bar{R}, \bar{\sigma}} V \cong W \bar{\otimes}_{\bar{P}, \bar{\nu}} (D \otimes_{R,\sigma} V)$ given by the trivial identification.

Proof. Note first that the relations (2.9), (2.10) and (2.11) may be written down respectively as

$$\begin{aligned}
&Q_W(v, w_P) \otimes Q_D(v, w_P) d_{P_R} \otimes Q_V(v, w_P)_R \\
&= Q_W(v_R, w)_P \otimes d_{R_P} Q_D(v_R, w) \otimes Q_V(v_R, w),
\end{aligned} \tag{2.12}$$

$$\begin{aligned}
&Q_W(v, \nu_1(w, w')) \otimes Q_D(v, \nu_1(w, w')) \nu_2(w, w')_R \otimes Q_V(v, \nu_1(w, w'))_R \\
&= \nu_1(Q_W(v, w), Q_W(Q_V(v, w), w')_P) \otimes \nu_2(Q_W(v, w), Q_W(Q_V(v, w), w')_P) \\
&\quad Q_D(v, w)_P Q_D(Q_V(v, w), w') \otimes Q_V(Q_V(v, w), w'),
\end{aligned} \tag{2.13}$$

$$\begin{aligned}
&Q_W(\sigma_2(v, v'), w)_P \otimes \sigma_1(v, v')_P Q_D(\sigma_2(v, v'), w) \otimes Q_V(\sigma_2(v, v'), w) \\
&= Q_W(v, Q_W(v', w)) \otimes Q_D(v, Q_W(v', w)) Q_D(v', w)_R \\
&\quad \sigma_1(Q_V(v, Q_W(v', w))_R, Q_V(v', w)) \otimes \sigma_2(Q_V(v, Q_W(v', w))_R, Q_V(v', w)),
\end{aligned} \tag{2.14}$$

for all $d \in D, v, v' \in V$ and $w, w' \in W$.

We will only prove (i) and (iii), while (ii) is similar to (i) and left to the reader.

Proof of (i):

The conditions (1.1) and (1.2) are very easy to prove and are left to the reader. We denote as usual by $R = r = \mathcal{R} = \bar{R}$ some more copies of R and by p another copy of P .

Proof of (1.3):

Let $v \in V, w, w' \in W$ and $d, d' \in D$; we compute:

$$\begin{aligned}
&(\mu_{W \bar{\otimes}_{P,\nu} D} \otimes id_V) \circ (id_{W \bar{\otimes}_{P,\nu} D} \otimes \bar{R}) \circ (\bar{R} \otimes id_{W \bar{\otimes}_{P,\nu} D})(v \otimes w \otimes d \otimes w' \otimes d') \\
&= (\mu_{W \bar{\otimes}_{P,\nu} D} \otimes id_V) \circ (id_{W \bar{\otimes}_{P,\nu} D} \otimes \bar{R})(Q_W(v, w) \otimes Q_D(v, w) d_R \otimes Q_V(v, w)_R \otimes w' \otimes d')
\end{aligned}$$

$$\begin{aligned}
&= (\mu_{W \otimes_{P, \nu} D} \otimes id_V)(Q_W(v, w) \otimes Q_D(v, w)d_R \otimes Q_W(Q_V(v, w)_R, w')) \\
&\quad \otimes Q_D(Q_V(v, w)_R, w')d'_r \otimes Q_V(Q_V(v, w)_R, w')_r) \\
&= \nu_1(Q_W(v, w), Q_W(Q_V(v, w)_R, w')_P) \otimes \nu_2(Q_W(v, w), Q_W(Q_V(v, w)_R, w')_P) \\
&\quad [Q_D(v, w)d_R]_P Q_D(Q_V(v, w)_R, w')d'_r \otimes Q_V(Q_V(v, w)_R, w')_r \\
&\stackrel{(2.6)}{=} \nu_1(Q_W(v, w), Q_W(Q_V(v, w)_R, w')_{P_p}) \otimes \nu_2(Q_W(v, w), Q_W(Q_V(v, w)_R, w')_{P_p}) \\
&\quad Q_D(v, w)_p d_{RP} Q_D(Q_V(v, w)_R, w')d'_r \otimes Q_V(Q_V(v, w)_R, w')_r \\
&\stackrel{(2.12)}{=} \nu_1(Q_W(v, w), Q_W(Q_V(v, w), w'_P)_p) \otimes \nu_2(Q_W(v, w), Q_W(Q_V(v, w), w'_P)_p) \\
&\quad Q_D(v, w)_p Q_D(Q_V(v, w), w'_P)d_{PR}d'_r \otimes Q_V(Q_V(v, w), w'_P)_{R_r} \\
&\stackrel{(2.13)}{=} Q_W(v, \nu_1(w, w'_P)) \otimes Q_D(v, \nu_1(w, w'_P))\nu_2(w, w'_P)\mathcal{R}d_{PR}d'_r \otimes Q_V(v, \nu_1(w, w'_P))\mathcal{R}_{R_r} \\
&\stackrel{(1.6)}{=} Q_W(v, \nu_1(w, w'_P)) \otimes Q_D(v, \nu_1(w, w'_P))[\nu_2(w, w'_P)d_P d']_R \otimes Q_V(v, \nu_1(w, w'_P))_R \\
&= \overline{R}(v \otimes \nu_1(w, w'_P) \otimes \nu_2(w, w'_P)d_P d') \\
&= \overline{R} \circ (id_V \otimes \mu_{W \otimes_{P, \nu} D})(v \otimes w \otimes d \otimes w' \otimes d'), \quad q.e.d.
\end{aligned}$$

Proof of (1.4):

For $v, v', v'' \in V$ we compute:

$$\begin{aligned}
&(\mu_{W \otimes_{P, \nu} D} \otimes id_V) \circ (id_{W \otimes_{P, \nu} D} \otimes \overline{\sigma}) \circ (\overline{R} \otimes id_V) \circ (id_V \otimes \overline{\sigma})(v \otimes v' \otimes v'') \\
&= (\mu_{W \otimes_{P, \nu} D} \otimes id_V) \circ (id_{W \otimes_{P, \nu} D} \otimes \overline{\sigma}) \circ (\overline{R} \otimes id_V)(v \otimes 1_W \otimes \sigma_1(v', v'') \otimes \sigma_2(v', v'')) \\
&= (\mu_{W \otimes_{P, \nu} D} \otimes id_V) \circ (id_{W \otimes_{P, \nu} D} \otimes \overline{\sigma})(Q_W(v, 1_W) \otimes Q_D(v, 1_W)\sigma_1(v', v'')_R \\
&\quad \otimes Q_V(v, 1_W)_R \otimes \sigma_2(v', v'')) \\
&= (\mu_{W \otimes_{P, \nu} D} \otimes id_V) \circ (id_{W \otimes_{P, \nu} D} \otimes \overline{\sigma})(1_W \otimes \sigma_1(v', v'')_R \otimes v_R \otimes \sigma_2(v', v'')) \\
&= (\mu_{W \otimes_{P, \nu} D} \otimes id_V)(1_W \otimes \sigma_1(v', v'')_R \otimes 1_W \otimes \sigma_1(v_R, \sigma_2(v', v'')) \otimes \sigma_2(v_R, \sigma_2(v', v''))) \\
&= 1_W \otimes \sigma_1(v', v'')_R \sigma_1(v_R, \sigma_2(v', v'')) \otimes \sigma_2(v_R, \sigma_2(v', v'')) \\
&\stackrel{(1.7)}{=} 1_W \otimes \sigma_1(v, v')\sigma_1(\sigma_2(v, v'), v'') \otimes \sigma_2(\sigma_2(v, v'), v'') \\
&= (\mu_{W \otimes_{P, \nu} D} \otimes id_V)(1_W \otimes \sigma_1(v, v') \otimes 1_W \otimes \sigma_1(\sigma_2(v, v'), v'') \otimes \sigma_2(\sigma_2(v, v'), v'')) \\
&= (\mu_{W \otimes_{P, \nu} D} \otimes id_V) \circ (id_{W \otimes_{P, \nu} D} \otimes \overline{\sigma})(1_W \otimes \sigma_1(v, v') \otimes \sigma_2(v, v') \otimes v'') \\
&= (\mu_{W \otimes_{P, \nu} D} \otimes id_V) \circ (id_{W \otimes_{P, \nu} D} \otimes \overline{\sigma}) \circ (\overline{\sigma} \otimes id_V)(v \otimes v' \otimes v''), \quad q.e.d.
\end{aligned}$$

Proof of (1.5):

For $v, v' \in V$, $w \in W$ and $d \in D$ we compute:

$$\begin{aligned}
&(\mu_{W \otimes_{P, \nu} D} \otimes id_V) \circ (id_{W \otimes_{P, \nu} D} \otimes \overline{\sigma}) \circ (\overline{R} \otimes id_V) \circ (id_V \otimes \overline{R})(v \otimes v' \otimes w \otimes d) \\
&= (\mu_{W \otimes_{P, \nu} D} \otimes id_V) \circ (id_{W \otimes_{P, \nu} D} \otimes \overline{\sigma}) \circ (\overline{R} \otimes id_V)(v \otimes Q_W(v', w) \\
&\quad \otimes Q_D(v', w)d_R \otimes Q_V(v', w)_R) \\
&= (\mu_{W \otimes_{P, \nu} D} \otimes id_V) \circ (id_{W \otimes_{P, \nu} D} \otimes \overline{\sigma})(Q_W(v, Q_W(v', w)) \\
&\quad \otimes Q_D(v, Q_W(v', w))[Q_D(v', w)d_R]_r \otimes Q_V(v, Q_W(v', w))_r \otimes Q_V(v', w)_R) \\
&= (\mu_{W \otimes_{P, \nu} D} \otimes id_V)(Q_W(v, Q_W(v', w)) \otimes Q_D(v, Q_W(v', w))[Q_D(v', w)d_R]_r \otimes 1_W \\
&\quad \otimes \sigma_1(Q_V(v, Q_W(v', w))_r, Q_V(v', w)_R) \otimes \sigma_2(Q_V(v, Q_W(v', w))_r, Q_V(v', w)_R))
\end{aligned}$$

$$\begin{aligned}
&= Q_W(v, Q_W(v', w)) \otimes Q_D(v, Q_W(v', w))[Q_D(v', w)d_R]_r \\
&\quad \sigma_1(Q_V(v, Q_W(v', w))_r, Q_V(v', w)_R) \otimes \sigma_2(Q_V(v, Q_W(v', w))_r, Q_V(v', w)_R)) \\
(1.6) \quad &\stackrel{=}{=} Q_W(v, Q_W(v', w)) \otimes Q_D(v, Q_W(v', w))Q_D(v', w)\mathcal{R}d_{R_r} \\
&\quad \sigma_1(Q_V(v, Q_W(v', w))_{\mathcal{R}_r}, Q_V(v', w)_R) \otimes \sigma_2(Q_V(v, Q_W(v', w))_{\mathcal{R}_r}, Q_V(v', w)_R) \\
(1.8) \quad &\stackrel{=}{=} Q_W(v, Q_W(v', w)) \otimes Q_D(v, Q_W(v', w))Q_D(v', w)\mathcal{R} \\
&\quad \sigma_1(Q_V(v, Q_W(v', w))_{\mathcal{R}}, Q_V(v', w))d_R \otimes \sigma_2(Q_V(v, Q_W(v', w))_{\mathcal{R}}, Q_V(v', w))_R \\
(2.14) \quad &\stackrel{=}{=} Q_W(\sigma_2(v, v'), w)_P \otimes \sigma_1(v, v')_P Q_D(\sigma_2(v, v'), w)d_R \otimes Q_V(\sigma_2(v, v'), w)_R \\
(2.2) \quad &\stackrel{=}{=} \nu_1(1_W, Q_W(\sigma_2(v, v'), w)_P) \otimes \nu_2(1_W, Q_W(\sigma_2(v, v'), w)_P)\sigma_1(v, v')_P \\
&\quad Q_D(\sigma_2(v, v'), w)d_R \otimes Q_V(\sigma_2(v, v'), w)_R \\
&= (\mu_{W \otimes_{P, \nu} D} \otimes id_V)(1_W \otimes \sigma_1(v, v') \otimes Q_W(\sigma_2(v, v'), w) \\
&\quad \otimes Q_D(\sigma_2(v, v'), w)d_R \otimes Q_V(\sigma_2(v, v'), w)_R) \\
&= (\mu_{W \otimes_{P, \nu} D} \otimes id_V) \circ (id_{W \otimes_{P, \nu} D} \otimes \overline{R})(1_W \otimes \sigma_1(v, v') \otimes \sigma_2(v, v') \otimes w \otimes d) \\
&= (\mu_{W \otimes_{P, \nu} D} \otimes id_V) \circ (id_{W \otimes_{P, \nu} D} \otimes \overline{R}) \circ (\overline{\sigma} \otimes id_{W \otimes_{P, \nu} D})(v \otimes v' \otimes w \otimes d), \quad q.e.d.
\end{aligned}$$

Proof of (iii):

The multiplication in $(W \otimes_{P, \nu} D) \otimes_{\overline{R}, \overline{\sigma}} V$ reads:

$$\begin{aligned}
&((w \otimes d) \otimes v)((w' \otimes d') \otimes v') \\
&= (w \otimes d)(w' \otimes d')_{\overline{R}} \overline{\sigma}_1(v_{\overline{R}}, v') \otimes \overline{\sigma}_2(v_{\overline{R}}, v') \\
&= (w \otimes d)(Q_W(v, w') \otimes Q_D(v, w')d'_R)_{\overline{R}} \overline{\sigma}_1(Q_V(v, w')_R, v') \otimes \overline{\sigma}_2(Q_V(v, w')_R, v') \\
&= (w \otimes d)(Q_W(v, w') \otimes Q_D(v, w')d'_R)(1_W \otimes \sigma_1(Q_V(v, w')_R, v')) \otimes \sigma_2(Q_V(v, w')_R, v') \\
&= (w \otimes d)(Q_W(v, w') \otimes Q_D(v, w')d'_R \sigma_1(Q_V(v, w')_R, v')) \otimes \sigma_2(Q_V(v, w')_R, v') \\
&= \nu_1(w, Q_W(v, w')_P) \otimes \nu_2(w, Q_W(v, w')_P)d_P Q_D(v, w')d'_R \sigma_1(Q_V(v, w')_R, v') \\
&\quad \otimes \sigma_2(Q_V(v, w')_R, v').
\end{aligned}$$

The multiplication in $W \otimes_{\overline{P}, \overline{\sigma}} (D \otimes_{R, \sigma} V)$ reads:

$$\begin{aligned}
&(w \otimes (d \otimes v))(w' \otimes (d' \otimes v')) \\
&= \overline{\nu}_1(w, w'_{\overline{P}}) \otimes \overline{\nu}_2(w, w'_{\overline{P}})(d \otimes v)_{\overline{P}}(d' \otimes v') \\
&= \overline{\nu}_1(w, Q_W(v, w')_P) \otimes \overline{\nu}_2(w, Q_W(v, w')_P)(d_P Q_D(v, w') \otimes Q_V(v, w'))(d' \otimes v') \\
&= \nu_1(w, Q_W(v, w')_P) \otimes (\nu_2(w, Q_W(v, w')_P) \otimes 1_V)(d_P Q_D(v, w') \otimes Q_V(v, w'))(d' \otimes v') \\
&= \nu_1(w, Q_W(v, w')_P) \otimes (\nu_2(w, Q_W(v, w')_P)d_P Q_D(v, w') \otimes Q_V(v, w'))(d' \otimes v') \\
&= \nu_1(w, Q_W(v, w')_P) \otimes \nu_2(w, Q_W(v, w')_P)d_P Q_D(v, w')d'_R \sigma_1(Q_V(v, w')_R, v') \\
&\quad \otimes \sigma_2(Q_V(v, w')_R, v'),
\end{aligned}$$

and we can see that the two multiplications are defined by the same formula. \square

Definition 2.4 *In the hypotheses of Theorem 2.3, the algebra structure on $W \otimes D \otimes V$ arising from (iii) in the theorem will be called the iterated crossed product afforded by the map Q .*

Theorem 2.3 admits the following converse:

Theorem 2.5 *Let $W \overline{\otimes}_{P,\nu} D$ and $D \otimes_{R,\sigma} V$ be two crossed products. Suppose that on $W \otimes D \otimes V$ we have an associative unital algebra structure (with multiplication denoted by \cdot) such that*

$$\begin{aligned} W \overline{\otimes}_{P,\nu} D &\rightarrow W \otimes D \otimes V, & w \otimes d &\mapsto w \otimes d \otimes 1_V, \\ D \otimes_{R,\sigma} V &\rightarrow W \otimes D \otimes V, & d \otimes v &\mapsto 1_W \otimes d \otimes v, \end{aligned}$$

are algebra maps (in particular, it follows that the unit of $W \otimes D \otimes V$ is $1_W \otimes 1_D \otimes 1_V$) and moreover, for all $w \in W$, $d \in D$, $v \in V$ we have

$$w \otimes d \otimes v = (w \otimes 1_D \otimes 1_V) \cdot (1_W \otimes d \otimes 1_V) \cdot (1_W \otimes 1_D \otimes v). \quad (2.15)$$

Then there exists a linear map $Q : V \otimes W \rightarrow W \otimes D \otimes V$ satisfying the hypotheses of Theorem 2.3 and such that the given algebra structure on $W \otimes D \otimes V$ coincides with the iterated crossed product afforded by Q . More precisely, the map Q is defined by

$$Q(v \otimes w) := (1_W \otimes 1_D \otimes v) \cdot (w \otimes 1_D \otimes 1_V), \quad \forall v \in V, w \in W. \quad (2.16)$$

Proof. We need to prove first that the map Q defined by formula (2.16) satisfies the conditions in Theorem 2.3. Since $1_W \otimes 1_D \otimes 1_V$ is the unit of $W \otimes D \otimes V$, it is clear that we have $Q(1_V \otimes w) = w \otimes 1_D \otimes 1_V$ and $Q(v \otimes 1_W) = 1_W \otimes 1_D \otimes v$, for all $v \in V$ and $w \in W$. We will denote $Q(v \otimes w) = (1_W \otimes 1_D \otimes v) \cdot (w \otimes 1_D \otimes 1_V) = Q_W(v, w) \otimes Q_D(v, w) \otimes Q_V(v, w)$. By regarding W , D , and V embedded canonically into $W \otimes D \otimes V$, we can regard elements w , d , v as elements in $W \otimes D \otimes V$; by (2.15) we can write $w \otimes d \otimes v = w \cdot d \cdot v$, for all $w \in W$, $d \in D$, $v \in V$ (in particular, we have $Q(v \otimes w) = v \cdot w = Q_W(v, w) \cdot Q_D(v, w) \cdot Q_V(v, w)$). Because of the algebra embeddings of $W \overline{\otimes}_{P,\nu} D$ and $D \otimes_{R,\sigma} V$ into $W \otimes D \otimes V$, we immediately obtain

$$d \cdot w = w_P \cdot d_P, \quad (2.17)$$

$$v \cdot d = d_R \cdot v_R, \quad (2.18)$$

$$w \cdot w' = \nu_1(w, w') \cdot \nu_2(w, w'), \quad (2.19)$$

$$v \cdot v' = \sigma_1(v, v') \cdot \sigma_2(v, v'), \quad (2.20)$$

for all $v, v' \in V$, $d \in D$, $w, w' \in W$. We prove now that Q satisfies the other conditions in Theorem 2.3, namely conditions (2.12)–(2.14).

Proof of (2.12):

$$\begin{aligned} &Q_W(v, w_P) \otimes Q_D(v, w_P) d_{P_R} \otimes Q_V(v, w_P)_R \\ &= Q_W(v, w_P) \cdot Q_D(v, w_P) \cdot d_{P_R} \cdot Q_V(v, w_P)_R \\ (2.18) \quad &\stackrel{=}{=} Q_W(v, w_P) \cdot Q_D(v, w_P) \cdot Q_V(v, w_P) \cdot d_P \\ &= Q(v \otimes w_P) \cdot d_P \\ &= v \cdot w_P \cdot d_P \\ (2.17) \quad &\stackrel{=}{=} v \cdot d \cdot w, \end{aligned}$$

$$\begin{aligned} &Q_W(v_R, w)_P \otimes d_{R_P} Q_D(v_R, w) \otimes Q_V(v_R, w) \\ &= Q_W(v_R, w)_P \cdot d_{R_P} \cdot Q_D(v_R, w) \cdot Q_V(v_R, w) \\ (2.17) \quad &\stackrel{=}{=} d_R \cdot Q_W(v_R, w) \cdot Q_D(v_R, w) \cdot Q_V(v_R, w) \end{aligned}$$

$$\begin{aligned}
&= d_R \cdot Q(v_R \otimes w) \\
&= d_R \cdot v_R \cdot w \\
(2.18) \quad &\stackrel{=}{=} v \cdot d \cdot w, \quad q.e.d.
\end{aligned}$$

Proof of (2.13):

$$\begin{aligned}
&Q_W(v, \nu_1(w, w')) \otimes Q_D(v, \nu_1(w, w')) \nu_2(w, w')_R \otimes Q_V(v, \nu_1(w, w'))_R \\
&= Q_W(v, \nu_1(w, w')) \cdot Q_D(v, \nu_1(w, w')) \cdot \nu_2(w, w')_R \cdot Q_V(v, \nu_1(w, w'))_R \\
(2.18) \quad &\stackrel{=}{=} Q_W(v, \nu_1(w, w')) \cdot Q_D(v, \nu_1(w, w')) \cdot Q_V(v, \nu_1(w, w')) \cdot \nu_2(w, w') \\
&= Q(v \otimes \nu_1(w, w')) \cdot \nu_2(w, w') \\
&= v \cdot \nu_1(w, w') \cdot \nu_2(w, w') \\
(2.19) \quad &\stackrel{=}{=} v \cdot w \cdot w', \\
&\nu_1(Q_W(v, w), Q_W(Q_V(v, w), w')_P) \otimes \nu_2(Q_W(v, w), Q_W(Q_V(v, w), w')_P) Q_D(v, w)_P \\
&\quad Q_D(Q_V(v, w), w') \otimes Q_V(Q_V(v, w), w') \\
&= \nu_1(Q_W(v, w), Q_W(Q_V(v, w), w')_P) \cdot \nu_2(Q_W(v, w), Q_W(Q_V(v, w), w')_P) \\
&\quad \cdot Q_D(v, w)_P \cdot Q_D(Q_V(v, w), w') \cdot Q_V(Q_V(v, w), w') \\
(2.19) \quad &\stackrel{=}{=} Q_W(v, w) \cdot Q_W(Q_V(v, w), w')_P \cdot Q_D(v, w)_P \cdot Q_D(Q_V(v, w), w') \cdot Q_V(Q_V(v, w), w') \\
(2.17) \quad &\stackrel{=}{=} Q_W(v, w) \cdot Q_D(v, w) \cdot Q_W(Q_V(v, w), w') \cdot Q_D(Q_V(v, w), w') \cdot Q_V(Q_V(v, w), w') \\
&= Q_W(v, w) \cdot Q_D(v, w) \cdot Q(Q_V(v, w) \otimes w') \\
&= Q_W(v, w) \cdot Q_D(v, w) \cdot Q_V(v, w) \cdot w' \\
&= Q(v \otimes w) \cdot w' \\
&= v \cdot w \cdot w', \quad q.e.d.
\end{aligned}$$

Proof of (2.14):

$$\begin{aligned}
&Q_W(\sigma_2(v, v'), w)_P \otimes \sigma_1(v, v')_P Q_D(\sigma_2(v, v'), w) \otimes Q_V(\sigma_2(v, v'), w) \\
&= Q_W(\sigma_2(v, v'), w)_P \cdot \sigma_1(v, v')_P \cdot Q_D(\sigma_2(v, v'), w) \cdot Q_V(\sigma_2(v, v'), w) \\
(2.17) \quad &\stackrel{=}{=} \sigma_1(v, v') \cdot Q_W(\sigma_2(v, v'), w) \cdot Q_D(\sigma_2(v, v'), w) \cdot Q_V(\sigma_2(v, v'), w) \\
&= \sigma_1(v, v') \cdot Q(\sigma_2(v, v') \otimes w) \\
&= \sigma_1(v, v') \cdot \sigma_2(v, v') \cdot w \\
(2.20) \quad &\stackrel{=}{=} v \cdot v' \cdot w, \\
&Q_W(v, Q_W(v', w)) \otimes Q_D(v, Q_W(v', w)) Q_D(v', w)_R \\
&\quad \sigma_1(Q_V(v, Q_W(v', w))_R, Q_V(v', w)) \otimes \sigma_2(Q_V(v, Q_W(v', w))_R, Q_V(v', w)) \\
&= Q_W(v, Q_W(v', w)) \cdot Q_D(v, Q_W(v', w)) \cdot Q_D(v', w)_R \\
&\quad \cdot \sigma_1(Q_V(v, Q_W(v', w))_R, Q_V(v', w)) \cdot \sigma_2(Q_V(v, Q_W(v', w))_R, Q_V(v', w)) \\
(2.20) \quad &\stackrel{=}{=} Q_W(v, Q_W(v', w)) \cdot Q_D(v, Q_W(v', w)) \cdot Q_D(v', w)_R
\end{aligned}$$

$$\begin{aligned}
& \cdot Q_V(v, Q_W(v', w))_R \cdot Q_V(v', w) \\
(2.18) \quad &= Q_W(v, Q_W(v', w)) \cdot Q_D(v, Q_W(v', w)) \cdot Q_V(v, Q_W(v', w)) \cdot Q_D(v', w) \cdot Q_V(v', w) \\
&= Q(v \otimes Q_W(v', w)) \cdot Q_D(v', w) \cdot Q_V(v', w) \\
&= v \cdot Q_W(v', w) \cdot Q_D(v', w) \cdot Q_V(v', w) \\
&= v \cdot Q(v' \otimes w) \\
&= v \cdot v' \cdot w, \quad q.e.d.
\end{aligned}$$

Hence, the hypotheses of Theorem 2.3 are satisfied, so we can consider the iterated crossed product afforded by the map Q . The only thing left to prove is that the original multiplication of $W \otimes D \otimes V$ coincides with the multiplication of the iterated crossed product (as it appears at the end of the proof of Theorem 2.3). So, we express the original multiplication of $W \otimes D \otimes V$ as follows:

$$\begin{aligned}
& (w \otimes d \otimes v) \cdot (w' \otimes d' \otimes v') \\
&= w \cdot d \cdot v \cdot w' \cdot d' \cdot v' \\
&= w \cdot d \cdot Q(v \otimes w') \cdot d' \cdot v' \\
&= w \cdot d \cdot Q_W(v, w') \cdot Q_D(v, w') \cdot Q_V(v, w') \cdot d' \cdot v' \\
(2.17), (2.18) \quad &\stackrel{=}{=} w \cdot Q_W(v, w')_P \cdot d_P \cdot Q_D(v, w') \cdot d'_R \cdot Q_V(v, w')_R \cdot v' \\
(2.19), (2.20) \quad &\stackrel{=}{=} \nu_1(w, Q_W(v, w')_P) \cdot \nu_2(w, Q_W(v, w')_P) d_P Q_D(v, w') d'_R \sigma_1(Q_V(v, w')_R, v') \\
&\quad \cdot \sigma_2(Q_V(v, w')_R, v') \\
&= \nu_1(w, Q_W(v, w')_P) \otimes \nu_2(w, Q_W(v, w')_P) d_P Q_D(v, w') d'_R \sigma_1(Q_V(v, w')_R, v') \\
&\quad \otimes \sigma_2(Q_V(v, w')_R, v'),
\end{aligned}$$

and this is exactly the multiplication of the iterated crossed product. \square

Example 2.6 We recall from [6] what was called there an iterated twisted tensor product of algebras. Let A, B, C be associative unital algebras, $R_1 : B \otimes A \rightarrow A \otimes B$, $R_2 : C \otimes B \rightarrow B \otimes C$, $R_3 : C \otimes A \rightarrow A \otimes C$ twisting maps satisfying the braid (or hexagon) equation

$$(id_A \otimes R_2) \circ (R_3 \otimes id_B) \circ (id_C \otimes R_1) = (R_1 \otimes id_C) \circ (id_B \otimes R_3) \circ (R_2 \otimes id_A).$$

Then we have an algebra structure on $A \otimes B \otimes C$ (called the iterated twisted tensor product) with unit $1_A \otimes 1_B \otimes 1_C$ and multiplication

$$(a \otimes b \otimes c)(a' \otimes b' \otimes c') = a(a'_{R_3})_{R_1} \otimes b_{R_1} b'_{R_2} \otimes (c_{R_3})_{R_2} c'.$$

If we consider the crossed products $A \overline{\otimes}_{P, \nu} B = A \otimes_{R_1} B$ and $B \otimes_{R, \sigma} C = B \otimes_{R_2} C$, where $P = R_1$, $R = R_2$, $\nu(a \otimes a') = aa' \otimes 1_B$ and $\sigma(c \otimes c') = 1_B \otimes cc'$, then one can easily see, by using Theorem 2.5, that the iterated twisted tensor product coincides with the iterated crossed product afforded by the map $Q : C \otimes A \rightarrow A \otimes B \otimes C$, $Q(c \otimes a) = a_{R_3} \otimes 1_B \otimes c_{R_3}$.

Example 2.7 We recall the following construction from [8]. Let H be a bialgebra, A (respectively B) a bialgebra in the category of left (respectively right) Yetter-Drinfeld H -modules, with module and comodule structures denoted by $H \otimes A \rightarrow A$, $h \otimes a \mapsto h \rightarrow a$, $A \rightarrow H \otimes A$, $a \mapsto a^1 \otimes a^2$, $B \otimes H \rightarrow B$, $b \otimes h \mapsto b \leftarrow h$, $B \rightarrow B \otimes H$, $b \mapsto b^1 \otimes b^2$, and comultiplications denoted

by $\Delta(a) = a_1 \otimes a_2$, $\Delta(b) = b_1 \otimes b_2$. Assume that we are given linear maps $\rightarrow: B \otimes A \rightarrow A$, $\leftarrow: B \otimes A \rightarrow B$ and $\sharp: B \otimes A \rightarrow H$, such that A is a left B -module via \rightarrow and B is a right A -module via \leftarrow and some more conditions (listed in [8], pages 39–40) are satisfied. Then by Theorem 3.3 in [8], $A \otimes H \otimes B$ becomes a bialgebra with the comultiplication of a two-sided cosmash product, unit $1_A \otimes 1_H \otimes 1_B$ and the following multiplication:

$$(a \otimes h \otimes b)(a' \otimes h' \otimes b') = a(h_1 \rightarrow (b_1^1 \rightarrow a'_1)) \otimes h_2 b_1^2 (b_2 \sharp a'_2) (a'_3)^1 h'_1 \otimes ((b_3 \leftarrow (a'_3)^2) \leftarrow h'_2) b'$$

This algebra structure is actually an iterated crossed product. Indeed, consider the crossed products $H \otimes_{R,\sigma} B = H \# B$ and $A \overline{\otimes}_{P,\nu} H = A \# H$, where $H \# B$ and $A \# H$ are respectively the usual right and left smash products (so we have $R(b \otimes h) = h_1 \otimes b \leftarrow h_2$, $\sigma(b \otimes b') = 1_H \otimes bb'$, $P(h \otimes a) = h_1 \rightarrow a \otimes h_2$, $\nu(a \otimes a') = aa' \otimes 1_H$). By using Theorem 2.5, it turns out that Sommerhäuser's algebra structure is the iterated crossed product afforded by the map

$$Q: B \otimes A \rightarrow A \otimes H \otimes B, \quad Q(b \otimes a) = b_1^1 \rightarrow a_1 \otimes b_1^2 (b_2 \sharp a_2) a_3^1 \otimes b_3 \leftarrow a_3^2.$$

Example 2.8 We recall several things about quasi-Hopf smash products (we use terminology and notation as in [2], [3]). Let H be a quasi-bialgebra and A (respectively B) a left (respectively right) H -module algebra. We can consider the left (respectively right) smash product $A \# H$ (respectively $H \# B$), which is an associative algebra having $A \otimes H$ (respectively $H \otimes B$) as underlying vector space, multiplication $(a \# h)(a' \# h') = (x^1 \cdot a)(x^2 h_1 \cdot a') \# x^3 h_2 h'$ (respectively $(h \# b)(h' \# b') = hh'_1 x^1 \# (b \cdot h'_2 x^2)(b' \cdot x^3)$) and unit $1_A \# 1_H$ (respectively $1_H \# 1_B$) (we denoted as usual $\Phi = X^1 \otimes X^2 \otimes X^3$ and $\Phi^{-1} = x^1 \otimes x^2 \otimes x^3$ the associator of H and its inverse). We can consider also the so-called two-sided smash product $A \# H \# B$, which is an associative algebra structure on $A \otimes H \otimes B$ with unit $1_A \otimes 1_H \otimes 1_B$ and multiplication

$$(a \# h \# b)(a' \# h' \# b') = (x^1 \cdot a)(x^2 h_1 y^1 \cdot a') \# x^3 h_2 y^2 h'_1 z^1 \# (b \cdot y^3 h'_2 z^2)(b' \cdot z^3),$$

where $\Phi^{-1} = y^1 \otimes y^2 \otimes y^3 = z^1 \otimes z^2 \otimes z^3$ are two more copies of Φ^{-1} .

One can see that $A \# H$ and $H \# B$ are crossed products, namely $A \# H = A \overline{\otimes}_{P,\nu} H$ and $H \# B = H \otimes_{R,\sigma} B$, where

$$\begin{aligned} P: H \otimes A &\rightarrow A \otimes H, & P(h \otimes a) &= h_1 \cdot a \otimes h_2, \\ \nu: A \otimes A &\rightarrow A \otimes H, & \nu(a \otimes a') &= (x^1 \cdot a)(x^2 \cdot a') \otimes x^3, \\ R: B \otimes H &\rightarrow H \otimes B, & R(b \otimes h) &= h_1 \otimes b \cdot h_2, \\ \sigma: B \otimes B &\rightarrow H \otimes B, & \sigma(b \otimes b') &= x^1 \otimes (b \cdot x^2)(b' \cdot x^3). \end{aligned}$$

An easy computation shows that for the two-sided smash product $A \# H \# B$ we have $(a \# h \# b) = (a \# 1_H \# 1_B)(1_A \# h \# 1_B)(1_A \# 1_H \# b)$, for all $a \in A$, $h \in H$ and $b \in B$, and that $A \# H \# B$ is the iterated crossed product obtained from $A \# H$ and $H \# B$ via the map $Q: B \otimes A \rightarrow A \otimes H \otimes B$, $Q(b \otimes a) = x^1 \cdot a \otimes x^2 \otimes b \cdot x^3$.

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